JKAU: Eng. Sc., **vol. 10 no.1**, pp. 87-105 (1418A.H. / 1998 A.D.)

A General Multi-Rigid Body Dynamic Modeling Procedure Applied to a Three Axes Mixed-Loop Base Driven Robot

MOHAMMAD H.F. DADO
Department of Mechanical Engineering,
Faculty of Engineering and Technology, University of Jordan,
Amman, Jordan

ABSTRACT. This paper introduces a general procedure for modeling the dynamic behavior of multi-rigid body systems having kinematic constraints between their various elements. This procedure is based on the principle of virtual work. Two sets of varying parameters are generated for the description of the kinematics of the system. These two sets are: The *dependent* and the *independent* coordinates. A set of constraint equations relates the two sets of coordinates. The virtual work principle is derived for the complete system using the dependent set of coordinates. The coordinate variation are related by the constraint equations leading to a reduced system of equations using only the selected independent degrees of freedom of the system.This procedure simplifies the modeling of large multi-rigid body systems subjected to kinematic constraints. This simplification in the procedure does not result in an increase in the number of equations of motion that defines the system's dynamics. A detailed numerical example, in which the features of the modeling procedure are emphasized, is presented. This numerical example introduces a spatial mixed loop robot with the unique feature of having all the driving motors *fixed* to the base of the robot.

1. Introduction

Multi-rigid body dynamics is an old area of research. Numerous publications are available in this area, which make it difficult to cover all the advances or different directions of progress. However, the two main approaches for the problem are the analytical dynamics approach, Kane and Levinson[1] presented a dynamic formulation based basically on the virtual work principle. They applied their formulation to open chains as in robotic manipulators[2]. Kane and Levinson's method was used by many researchers, Wang[3] and Huston[4], where it is applied to systems with kinematic and force constraints. The virtual power method which is another form of the virtual work approach was used by Garcia *et al.*[5]. The Lagrangian approach is analytical dynamics was used extensively by many researchers specially in the area of robotic manipulators, Uicker[6], Paul[7], Lewis[8], and Agarwal and Chung[9]. Luh, *et al.*[10] presented an efficient formulation for a recursive computational scheme based on the Newton-Euler approach for serial manipulators. Although their scheme is applicable for open chain systems it becomes inefficient for closed or mixed chain systems.

The scheme presented in this paper is based on the virtual work approach, which is the basis to all the analytical dynamics formulation schemes. It deals with two types of coordinates: *Dependent* and *independent* coordinates. These two sets of coordinates facilitate the systematic generation of the body's dynamic matrix and vector using the dependent coordinates. The body's dynamic matrix and vector are derived through the manipulation of the body's linear and angular displacement variation transformation matrices. These matrices are similar in concept to the partial linear and angular velocities of Kane's and Levinson's[1] formulation. The kinematic constraint equations which relate the dependent and independent sets of coordinates impose what could be described as the boundary conditions between the various body dynamic matrices and vectors. This will lead to the generation of a number of independent equations of motion equal to the number of independent coordinates. In the Lagrangian approach, if the chosen generalized coordinates are not independent, the number of equations describing the motion of the system is increased to accommodate the constraint equations, Ginsberg[11]. Therefore, if the number of independent degrees of freedom of the system is n and the total number of the chosen generalized coordinates is $m+n$, then the number of constraint equations is m leading to a total of $2m+n$ equations of motion. Although this type of formulation might be required in certain situations, the main interest usually is in relating the generalized coordinates with the applied forces and torques. The dynamic modeling procedure developed in this paper limits the number of equations of motion to the number of independent generalized coordinates of the system regardless to the number of kinematic constraint equations.

A numerical example is presented where a three axes spatial robot is dynamically modeled. The spatial robot introduced in this example has a unique important feature. The three driving motors in this robot are completely fixed to the base of the robot. The motion is transmitted using planetary and open gear drives and a pantograph mixed loop configuration. By having the driving motors fixed at the base, the dynamic loading on the robot links is reduced resulting in a smooth dynamic behavior specially reducing the vibrational component of motion, which is not considered in this paper. The dynamic equations of motion derived for this robot are used to implement a position control strategy. The equations of motion are integrated in order to simulate the dynamic response of the robot corresponding to be implemented control strategy. The steps of the modeling procedure are explained using this mixed loop robot shown in Fig. 1. The following section introduces the mixed loop robot considered in this paper and defines its configuration and its various elements.

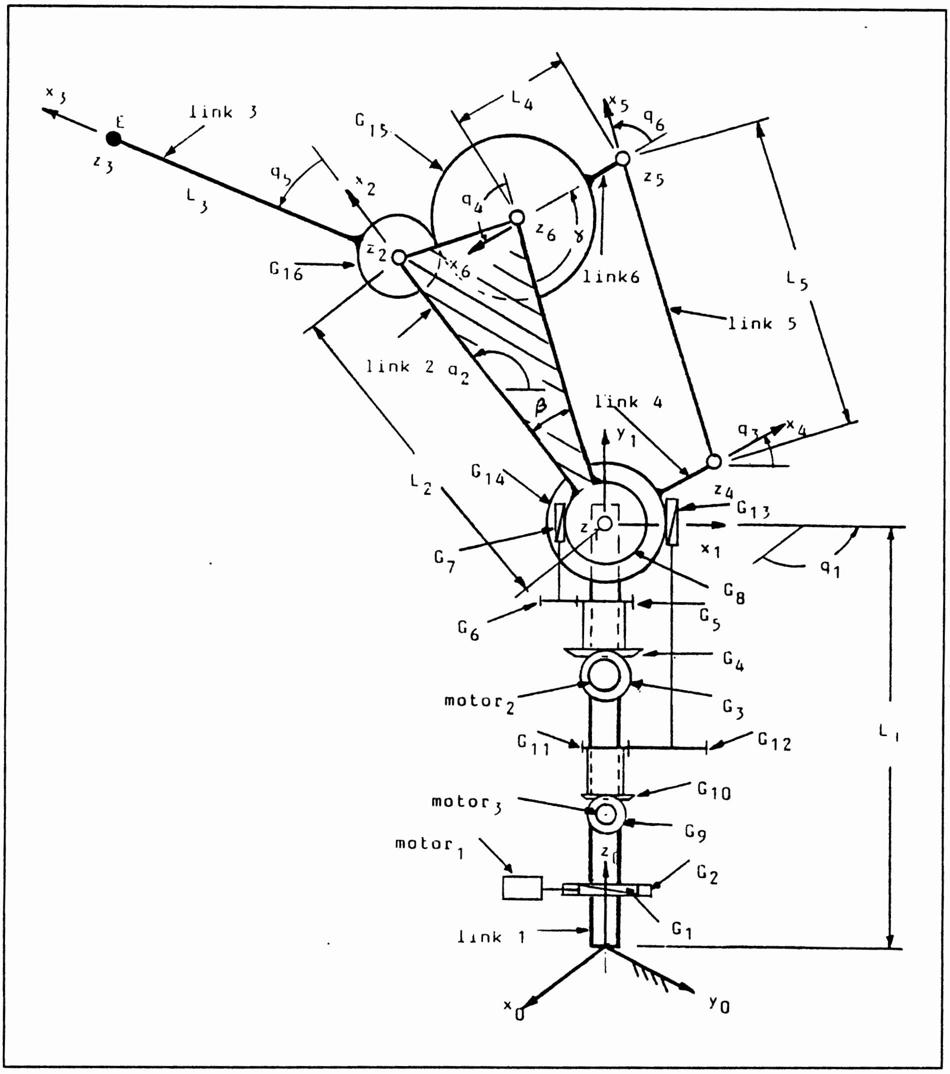

FIG. 1. A schematic drawing of the mixed loop robot with all parameters shown.

2. The Mixed Loop Robot

The mixed loop robot shown in Fig. 1 is used to explain the steps and to emphasize the various features of the dynamic modeling procedure presented in this paper. The robot consists of seven links including the ground link, where each link has its local coordinate system attached to it as indicated in Fig. 1. The Denavit and Hartenberg (D-H)[12] approach of assigning coordinate systems is used. For the robot shown in Fig. 1, there are six homogeneous transformation matrices relating the different links of the mixed loop robot..These matrices are defined using the kinematic parameters of the robot as indicated in Appendices A and B. The three driving motors are $motor_1$, $motor_2$ and $motor_3$. They are fixed at the based and their motions are transmitted to the links of the robot through gears and shafts as shown in Fig. 1. The worm gear G_1 is attached to $motor_1$ and drives worm wheel G_2 which is attached to link 1 resulting in its rotation about the axis Z_0. Bevel gear G_3 is attached to $motor_2$ and drives worm wheel G_8 through the gears G_5, G_6 and G_7. This planetary gear drive combination results in the rotation of link 2 about the Z_1 axis. Similarly, gears G_9 through G_{14} provide the drive train between $motor_3$ and link 4 which rotates independently about the Z_1 axis. Link 4 drives link 6 through link 5 using the pantograph configuration. Gear G_{15} which is attached to link 6 drives gear G_{16} resulting in the rotation of link 3 about the Z_2 axis. The various constant kinematic parameters and the inertial properties of the links are given in Appendix B.

3. The Modeling Procedure

This procedure utilizes the D-H homogeneous transformation matrices in defining the links' kinematic parameters and in coordinate transformations between links coordinates systems. The modeling procedure consists of six modeling steps leading to the equations of motion of a multi-rigid body system. The following subsections explain the modeling steps and apply them to the mixed loop robot.

3.1 Dependent and Independent Coordinates Assignment

The generalized coordinates that define the varying kinematic parameters of the system are divided into two types: Dependent and independent coordinates. The constraint equations define the dependent coordinates in terms of the independent coordinates. Naturally, the independent coordinates are those associated with the independent drives or inputs of the system. Consequently, all other coordinates are dependent coordinates. This feature of the modeling procedure gives flexibility in generalized coordinate assignment and in defining the actual drives of the dynamic system. For the mixed loop robot, the variables ψ_1, ψ_2, and ψ_3 are the independent coordinates and they correspond to the angular rotations of the output shafts of $motor_1$, $motor_2$, and $motor_3$, respectively. The variables $q_1, q_2, \ldots, q_6$ are the dependent coordinates and they define the varying parameters in the homogeneous matrices $[A]_0^1, [A]_1^2, \ldots, [A]^i$ as indicated in Table B-1 of Appendix B.

3.2 The Constraint Equations

The equations that define the dependent coordinates in terms of the independent coordinates are called the constraint equations. The expressions of the first order variations of the dependent coordinates in terms of the first order variations of the independent coordinates are required in order to generate the equations of motion. In addition, the second time derivatives of the dependent coordinates need to be expressed in terms of the independent coordinates and their first and second order time derivatives. In general, the dependent coordinates vector $\{Q\}$ can be expressed in terms of the independent coordinates vector as $\{\Psi\}$ as

$$\{Q\} = \{F_c(\{\Psi\}, t)\} \tag{1}$$

The first order variation of the dependent coordinates vector $\{\delta Q\}$ is obtained by expressing the total differential of $\{F_c(\{\Psi\}, t\}$ as

$$\{\delta Q\}_{m\times 1} = [C]_{m\times n}\{\delta\Psi\}_{n\times 1} + \delta t\left\{\frac{\partial F_c}{\partial t}\right\}_{m\times 1} \quad (2)$$

The elements of the matrix $[C]$ are the partial derivatives of the vector of the constraint functions $\{F_c\}$ with respect to the independent coordinates and they are given by

$$c_{ij} = \frac{\partial f_{ic}}{\partial \psi_j} \quad (3)$$

where f_{ic} is the $i-th$ constraint equation and ψ_j is the $j-th$ independent coordinate in the independent coordinates vector $\{\Psi\}$. The second time derivative of the dependent coordinates vector $\{\ddot{Q}\}$is expressed as

$$\{\ddot{Q}\} = [C]\{\ddot{\Psi}\} + [\dot{C}]\{\dot{\Psi}\} + \left\{\frac{\partial^2 F_c}{\partial t^2}\right\} \quad (4)$$

The elements of $[\dot{C}]$ are the total time derivatives of c_{ij}. The constraint equations for the mixed loop robot are defined in Appendix B.

3.3 Defining the Position and Orientation Variations

The first order variations of the position of the center of mass of every rigid body and the variations in the orientation of its coordinate system are defined in terms of the first order variations of the dependent coordinates. These definitions are obtained through the differentiation of the expression defining the position of the center of mass of the rigid body in the intertial coordinates. The position of the center of mass of body i is given by

$$\{P_i\}_o = [T]_o^i\{P_i\}_i \quad (5)$$

where

$$[T]_o^i = [\hat{A}]_{i_1}[\hat{A}]_{i_2}...[\hat{A}]_{i_{k_i}} \quad (6)$$

$\{P_i\}_o$ and $\{P_i\}_i$ are the position of the center of mass of the $i-th$ rigid body in the inertial frame and in the body's local coordinate system, respectively. The matrix $[\hat{A}]_{i_j}$ is the $j-th$ homogeneous transformation matrix in the sequence of transformation matrices between the inertial reference frame and the coordinate system associated with the $i-th$ rigid body and k_i is the number of matrices in the sequence. The matrix $[T]_o^i$ transforms the coordinates of a point given in the $i-th$ rigid body coordinate system into the inertial coordinate system.

In the mixed loop robot example, consider the rigid body corresponding to link 6. The sequence of transformation matrices between the inertial frame and the local coordinate system of link 6 is $[A]_0^1[A]_1^4[A]_4^5[A]_5^6 = [T]_o^6$. The corresponding dependent variables associated with this sequence are q_1, q_3, q_6, and q_4, (see Appendix B). Note that, for example, $[A]_1^4$ corresponds to $[\hat{A}]_{6_2}$ in the sequence of transformation matrices as defined by Equation (6). In addition, note that q_3 corresponds to $\hat{q}_{6_2}$which denotes the varying coordinate in $[\hat{A}]_{6_2}$. The total number of matrices in the sequence of link 6 is $k_6 = 4$.

The variations of the position of the center of mass can be expressed as

$$\begin{Bmatrix} \delta p_{x_i} \\ \delta p_{y_i} \\ \delta p_{z_i} \\ 0 \end{Bmatrix}_o = \left[\sum_{j=1}^{k_i} \delta \hat{q}_{i_j} [V]_{i_j} \right] \begin{Bmatrix} p_{x_i} \\ p_{y_i} \\ p_{z_i} \\ 1 \end{Bmatrix}_i \tag{7}$$

where

$$[V]_{i_j} = [\hat{A}]_{i_1} [\hat{A}]_{i_2} \ldots [\hat{B}]_{i_j} [\hat{A}]_{i_j} \ldots [\hat{A}]_{i_k} \tag{8}$$

$[\delta p_{x_i}\, \delta p_{y_i}\, \delta p_{z_i}]_o^T$ is the variations vector in the position of the center of mass of the $i-th$ rigid body expressed in the inertial reference frame. The variable $\hat{q}_{i_j}$ is the varying parameter associated with the transformation matrix $[\hat{A}]_{i_j}$ and corresponds to one of the dependent variables in the vector Q. The matrix $[\hat{B}]_{i_j}$ is the differential operator matrix and it depends on the type of kinematic pair associated with the transformation matrix as illustrated in Appendix A. The position variations can be expressed in compact form as

$$\begin{Bmatrix} \delta p_{x_i} \\ \delta p_{y_i} \\ \delta p_{z_i} \end{Bmatrix}_o = \begin{bmatrix} D_{x_{i1}} & \ldots & D_{x_{im}} \\ D_{y_{i1}} & \ldots & D_{y_{im}} \\ D_{z_{i1}} & \ldots & D_{z_{im}} \end{bmatrix} \begin{Bmatrix} \delta q_1 \\ \vdots \\ \delta q_m \end{Bmatrix} \tag{9}$$

The matrix $[D]_{i_{3\times m}}$ in Equation (9) is called the linear displacement variation transformation matrix of the $i-th$ rigid body. It should be noted here that the position variations of any point in the rigid body other than its center of mass could be obtained in the same manner by just changing the position vector $\{P_i\}_i$ in Equation (7). Other points of interest on a rigid body are points of load application as the case in the end effector of a robot.

The variations in the orientation of the coordinate system of the $i-th$ rigid body need to be expressed in terms of the dependent coordinates variations. These orientations variations are expressed in the local coordinate system in order to deal with the constant local inertial properties of the rigid body. These variations are given by

$$\begin{bmatrix} 0 & -\delta\phi_{z_i} & \delta\phi_{y_i} \\ \delta\phi_{z_i} & 0 & -\delta\phi_{x_i} \\ -\delta\phi_{y_i} & \delta\phi_{x_i} & 0 \end{bmatrix}_i = \left[[\tilde{T}]_o^i\right]^T \left[\sum_{j=1}^{k_i} \delta\hat{q}_{i_j} [\tilde{V}]_{i_j} \right] \tag{10}$$

where $\delta\phi_{x_i}$, $\delta\phi_{z_i}$, and $\delta\phi_{z_i}$ are the variations in the orientation of the rigid body i about its local x, y, and z axes, respectively. $[\tilde{T}]_o^i$ and $[\tilde{V}]_{i_j}$ are the upper 3×3 submatrices of the respective $[T]_o^i$ and $[V]_{i_j}$ matrices and they correspond to the orientation matrices. Rearranging Equation (10), the variations of the rigid body orientation can be written in terms of the dependent coordinates variations as

$$\begin{Bmatrix} \delta\phi_{x_i} \\ \delta\phi_{y_i} \\ \delta\phi_{z_i} \end{Bmatrix}_i = \begin{bmatrix} O_{x_{i1}} & \cdots & O_{x_{im}} \\ O_{y_{i1}} & \cdots & O_{y_{im}} \\ O_{z_{i1}} & \cdots & O_{z_{im}} \end{bmatrix} \begin{Bmatrix} \delta q_1 \\ \vdots \\ \delta q_m \end{Bmatrix} \tag{11}$$

The matrix $[Q]_{i_{3\times m}}$ in Equation (11) is called the angular displacement variation transformation matrix of the $i-th$ rigid body.

3.4 Linear and Angular Acceleration

The linear and angular accelerations of the $i-th$ rigid body are obtained by taking the second order time derivative of the transformation matrix $[T]_o^i$ given by Equation (6). The resulting expression is broken into two terms, one is associated with the acceleration vector of the dependent coordinates and the other is associated with velocity vector product terms. This separation is done in order to allow for the integration of the equations of motion. The second order time derivative of $[T]_o^i$ is given by

$$[\ddot{T}]_o^i = \left[\sum_{j=1}^{k_i} \ddot{\hat{q}}_{i_j} [V]_{i_j} \right] + \left[\sum_{j=1}^{k_1} \dot{\hat{q}}_{i_j} [\dot{V}]_{i_j} \right] \tag{12}$$

where

$$[\dot{V}]_{i_j} = \left[\sum_{l=1, l\neq j}^{k_i} \dot{\hat{q}}_{i_l} [\hat{A}]_{i_1} \cdots [\hat{B}]_{i_l} [\hat{A}]_{i_l} \cdots [\hat{B}]_{i_j} [\hat{A}]_{i_j} \cdots [\hat{A}]_{i_k} \right] \tag{13}$$
$$+ \dot{\hat{q}}_{i_j} [\hat{A}]_{i_1} \cdots [\hat{B}]_{i_j}^2 [\hat{A}]_{i_j} \cdots [\hat{A}]_{i_k}$$

The linear and angular accelerations of rigid body i could be written after simple manipulations and the use of Equations (8-13) as

$$\{\ddot{P}\}_i = [D]_i \{\ddot{Q}\} + \{A_p\}_i \tag{14}$$

$$\{\ddot{\Phi}\}_i = [O]_i \{\ddot{Q}\} + \{A_o\}_i \tag{15}$$

The matrices $[D]_i$ and $[O]_i$ are the same displacement variations transformation matrices presented in Equations (9) and (11). The vectors $\{A_p\}_i$ and $\{A_o\}_i$ are generated from the second right hand term of Equation (12). They are given by

$$\begin{Bmatrix} \{A_p\}_i \\ 0 \end{Bmatrix} = \left[\sum_{j=1}^{k_i} \dot{\hat{q}}_{i_j} [\dot{V}]_{i_j} \right] \begin{Bmatrix} p_{x_i} \\ p_{y_i} \\ p_{z_i} \\ 1 \end{Bmatrix}_i \tag{16}$$

and

$$\{A_o\}_i = \begin{Bmatrix} A_{ox} \\ A_{oy} \\ A_{oz} \end{Bmatrix}_i \tag{17}$$

where

$$\begin{bmatrix} 0 & -A_{oz} & A_{oy} \\ A_{oz} & 0 & -A_{ox} \\ -A_{oy} & A_{ox} & 0 \end{bmatrix}_i = \left[[\tilde{T}]_o^i\right]^T \left[\sum_{j=1}^{k_i} \dot{\hat{q}}_{i_j} [\dot{\tilde{V}}]_{i_j} \right] - [\Omega]_i^2 \tag{18}$$

The matrix $[\Omega]_i$ is the angular velocity matrix expressed in the local coordinate system of the $i-th$ body. It is given by

$$[\Omega]_i = \begin{bmatrix} 0 & -\dot{\phi}_{z_i} & \dot{\phi}_{y_i} \\ \dot{\phi}_{z_i} & 0 & -\dot{\phi}_{x_i} \\ -\dot{\phi}_{y_i} & \dot{\phi}_{x_i} & 0 \end{bmatrix}_i \tag{19}$$

where

$$\{\dot{\Phi}\} = [O]_i \{\dot{Q}\} \tag{20}$$

3.5 Unconstrained Rigid Body Dynamic Matrix and Vector

The virtual work principle states that the total work done during an infinitesimal displacement of any one of the independent degrees of freedom of the system is equal to zero. This total work consists of the work done by the applied forces and torques, the inertial loads, and the force or torque applied in the direction of the degree of freedom itself. For rigid body *i,* the virtual work done by the inertial loads and any applied loads can be expressed in terms of the virtual displacements of the dependent coordinates as

$$\begin{aligned} \delta W_{b_i} = & -[F_{x_{I_i}} F_{y_{I_i}} F_{z_{I_i}}]_o \begin{Bmatrix} \delta p_{x_i} \\ \delta p_{y_i} \\ \delta p_{z_i} \end{Bmatrix}_o \\ & -[M_{x_{I_i}} M_{y_{I_i}} M_{z_{I_i}}]_i \begin{Bmatrix} \delta \phi_{x_i} \\ \delta \phi_{y_i} \\ \delta \phi_{z_i} \end{Bmatrix}_i \\ & +[m_i g_x \; m_i g_y \; m_i g_z]_o \begin{Bmatrix} \delta p_{x_i} \\ \delta p_{y_i} \\ \delta p_{z_i} \end{Bmatrix}_o \\ & + \left[\sum_{l=1}^{n_{ief}} [F_{x_{Eil}} \; F_{y_{Eil}} \; F_{z_{Eil}}]_o \begin{Bmatrix} \delta p_{x_{Eil}} \\ \delta p_{y_{Eil}} \\ \delta p_{z_{Eil}} \end{Bmatrix}_o \right] \\ & + \left[\sum_{l=1}^{n_{iem}} [M_{x_{Eil}} \; M_{y_{Eil}} \; M_{z_{Eil}}]_i \begin{Bmatrix} \delta \phi_{x_i} \\ \delta \phi_{y_i} \\ \delta \phi_{z_i} \end{Bmatrix}_i \right] \end{aligned} \tag{21}$$

where

$[F_{x_{I_i}} F_{y_{I_i}} F_{z_{I_i}}]_o^T$ is the inertial force vector at the center of mass of rigid body i expressed in the inertial coordinate system,

$[M_{x_{I_i}} M_{y_{I_i}} M_{z_{I_i}}]_i^T$ is the inertial moment vector of rigid body i expressed in the local coordinate system.

m_i is the mass of rigid body i,

g_x, g_y, and g_z are the components of the acceleration of gravity in the inertial coordinate system,

$[F_{x_{Eil}} F_{y_{Eil}} F_{z_{Eil}}]_o^T$ is the vector of externally applied forces at point l of rigid body i and it is expressed in the inertial coordinate system,

$[\delta p_{x_{Eil}} \delta p_{y_{Eil}} \delta p_{z_{Eil}}]_o^T$ is the linear displacement variations of point l in rigid body i,

$[M_{x_{Eil}} M_{y_{Eil}} M_{z_{Eil}}]_i^T$ is the $l-th$ externally applied moment vector on rigid body i, and is expressed in the local coordinate system, and

n_{ief} and n_{iem} is the number of externally applied forces and moments on rigid body i, respectively.

The inertial force vector $\{F_{I_i}\}_o = [F_{x_{I_i}} F_{y_{I_i}} F_{z_{I_i}}]_o^T$ can be written using Equation (14) in terms of the velocity and acceleration of the dependent coordinates and the inertial properties as

$$\{F_{I_i}\}_o = m_i\left[[D]_i\{\ddot{Q}\} + \{A_p\}_i\right] \tag{22}$$

The inertial moment vector $\{M_{I_i}\}_i = [M_{x_{I_i}} M_{y_{I_i}} M_{z_{I_i}}]_i^T$ can be similarly written using Equation (15) as

$$\{M_{I_i}\}_i = [I_m]_i\left[[O]_i\{\ddot{Q}\} + \{A_o\}_i\right] + [\Omega]_i[I_m]_i\{\dot{\Phi}\} \tag{23}$$

where $[I_m]_i$ is the mass moment of inertia matrix of rigid body i expressed in the local coordinate system. Substituting Equations (22) and (23) in Equation (21) and using Equations (9) and (11), the virtual work done by the inertial and applied loads on rigid body i can be written in terms of the dependent coordinates variations as

$$\delta W_{bi} = \left[\{\ddot{Q}\}^T[DD_i]_{m\times m} + [AD_i]_{1\times m}\right]\{\delta Q\}_{m\times 1} \tag{24}$$

where

$$[DD_i]_{m\times m} = -m_i[D]_i^T[D]_i - [O]_i^T[I_m]_i^T[O]_i \tag{25}$$

$$\{AD_i\}^T = -m_i\{A_p\}_i^T[D]_i - \{A_o\}_i^T[I_m]_i^T[O]_i - \left[[\Omega][I_m]_i\{\dot{\Phi}\}\right]^T[O]_i + [AE_i]_{1\times m} \tag{26}$$

The one row matrix $[AE_i]_{1\times m}$ corresponds to the gravity and externally applied forces and moments of rigid body i and it is derived from the last three right hand terms in Equation (21).

The matrix $[DD_i]_{m\times m}$ and the vector $\{AD_i\}$ are called the unconstrained rigid body dynamic matrix and vector. They are computed individually for each rigid body regardless to the constraint equations or the actual independent degrees of freedom of the system. With this property, the rigid body dynamic matrix and vector are analogous of the element stiffness matrix and load factor of the finite element method.

3.6 Equations of Motion

The final set of the equations of motion is obtained by adding up the dynamic matrices and vectors of all the rigid bodies in the system and imposing the constraint equations. The total virtual work done by the inertial and external loads of M rigid bodies is given by

$$\delta W_{RB} = \sum_{i=1}^{M} \delta W_{bi} = \left[\{\ddot{Q}\}^T \left[\sum_{i=1}^{M} [D\,D_i]_{m\times m} \right] + \left[\sum_{i=1}^{M} [A\,D_i]_{1\times m} \right] \right] \{\delta Q\}_{m\times 1} \tag{27}$$

Imposing the constraint Equations (1), (2), and (4) on Equation (27) leads to

$$\begin{aligned} \delta W_{RB} = \Bigg[& \{\ddot{\Psi}\}^T [C]^T \left[\sum_{i=1}^{M} [D\,D_i]_{m\times m} \right] + \left[\sum_{i=1}^{M} [A\,D_i]_{1\times m} \right] + \\ & \left[[\dot{C}]\{\dot{\Psi}\} + \left\{ \frac{\partial^2 F_c}{\partial t^2} \right\} \right]^T \left[\sum_{i=1}^{M} [D\,D_i]_{m\times m} \right] \Bigg] [C]\{\delta \Psi\}_{n\times 1} \end{aligned} \tag{28}$$

The total virtual work done by conservative forces and moments for all the virtual displacements that are consistent with the system's constraints must be zero. This total work consists of the work done by the inertial and external loads of the rigid bodies δW_{RB} and the work done by the driving loads. This later work is expressed as

$$\delta W_D = [\tau_1 \tau_2 \cdots \tau_n] \begin{Bmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \vdots \\ \delta\psi_n \end{Bmatrix} \tag{29}$$

where τ_k is the driving load (force or torque) associated with the $k-th$ independent degree of freedom. Combining Equations (28) and (29) and the virtual work principle, the equations of motion for a multi-rigid body system are written as

$$[D\,M(\Psi)]\{\ddot{\Psi}\} + \{FE(\Psi,\dot{\Psi})\} + \{\tau\} = \{0\} \tag{30}$$

where $[DM(\Psi)]$ is the system's inertia matrix and it is given by

$$[D\,M(\Psi)] = [C]^T \left[\sum_{i=1}^{M} [D\,D_i]_{m\times m} \right]^T [C] \tag{31}$$

and $\{FE(\Psi,\dot{\Psi})\}$ is the system's Coriolis, centrifugal, and externally applied loads vector and it is given by

$$\{F'E(\Psi,\dot{\Psi})\} = [C]^T \left[\sum_{i=1}^{M} [A\,D_i]_{1\times m} \right] + [C]^T \left[\sum_{i=1}^{M} [D\,D_i]_{m\times m} \right]^T \left[[\dot{C}]\{\dot{\Psi}\} + \left\{ \frac{\partial^2 F_c}{\partial t^2} \right\} \right] \tag{32}$$

A computer program is developed to generate the dynamic equations of motion for a multi-rigid body system. The input to this program consists of the following:

- The parameters of the homogeneous transformation matrices that describe the kinematic parameters of the rigid bodies and the connecting kinematic pairs. Each of these matrices is given an identification number and assigned a coordinate number that corresponds to the varying parameter of the matrix.
- For each rigid body:
  1. The numbers of the transformation matrices sequence that lies in the path between the rigid body and the internal reference frame.
  2. The inertial properties for each rigid body.
- A subroutine that contains the constraint equations and their derivatives.

4. The Numerical Example

The equations of motion of the mixed loop robot presented in Section 2 are derived using the procedure explained in Section 3. The equations of motion are solved in the forward dynamic mode, where the driving torques of the motors are defined using a specified control law. The end effector of the mixed loop robot (point *E*) tracks a specific path in space with a programmed motion where the displacement, velocity, and acceleration along the path are given functions of time. Through inverse kinematic analysis, the desired independent coordinates positions, velocities and accelerations are determined[13]. These desired values along with the inertial properties of the robot are used in the control law to determine the driving torques. The following subsections define the control law used in driving the mixed loop robot and the programmed path tracking of the end effector.

4.1 The Control Law

The control law used in driving the mixed loop robot is a joint PD (Proportional-Derivative) control plus partial feed-forward torque. The feed-forward torque part consists only from the driving torque due to the desired independent coordinate accelerations. This control law could be stated as[14]

$$\{\tau_d\} = [K_p]\{\Psi_d - \Psi_a\} + [K_v]\{\dot{\Psi}_d - \dot{\Psi}_a\} + [DM(\Psi_a)]\{\ddot{\Psi}_d\} \qquad (33)$$

where

- $\{\tau_d\}$ is the control torque vector,
- $[K_p]$ is the diagonal position gains matrix,
- $[K_v]$ is the diagonal velocity gains matrix,
- $\{\Psi_d\}$ is the desired position vector of the independent driven coordinates, and
- $\{\Psi_a\}$ is the actual position vector of the independent driven coordinates.

4.2 Programmed End Effector Path Tracking

The desired path of the end effector is chosen to be a straight line in space. The displacement time function along the path is the cycloidal motion program which is given by

$$s = S_T\left(\frac{t}{T_T} - \frac{1}{2\pi}\sin\left(\frac{2\pi t}{T_T}\right)\right) \tag{34}$$

where

s is the position along the path at time t,

S_T is the total length of the path, and

T_T is the total specified time to track the path.

This motion program has the advantage of having zero velocities and accelerations at the beginning and end of motion.

The control law and the path tracking parameters for the numerical example along with the load data are given in Appendix B.

4.3 Results and Discussion

The dynamic equations of motion for the mixed loop robot are solved in the forward dynamic mode. The driving torques are specified by the control law where the aim is to force the end effector to track the specified spatial path. The equations of motion are solved using the fourth order Runge-Kutta method. Figures 2, 3, and 4 compare the desired and actual values of the position, velocity, and acceleration of the end effector. Figure 5 shows the time history of the three driving motor torques.

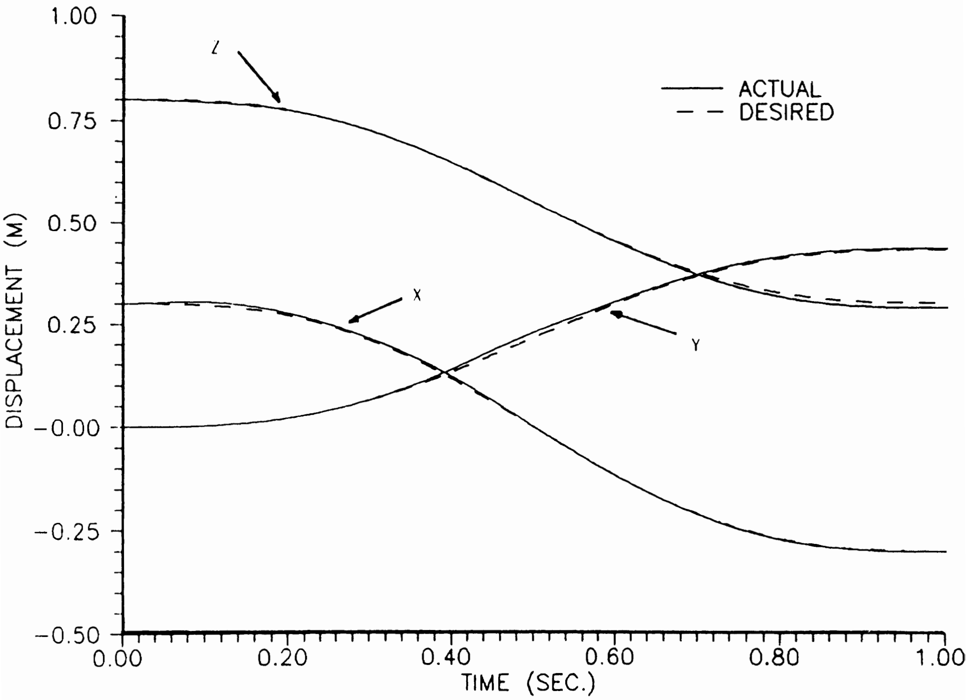

FIG. 2. The actual and desired displacement time history of the end effector point E.

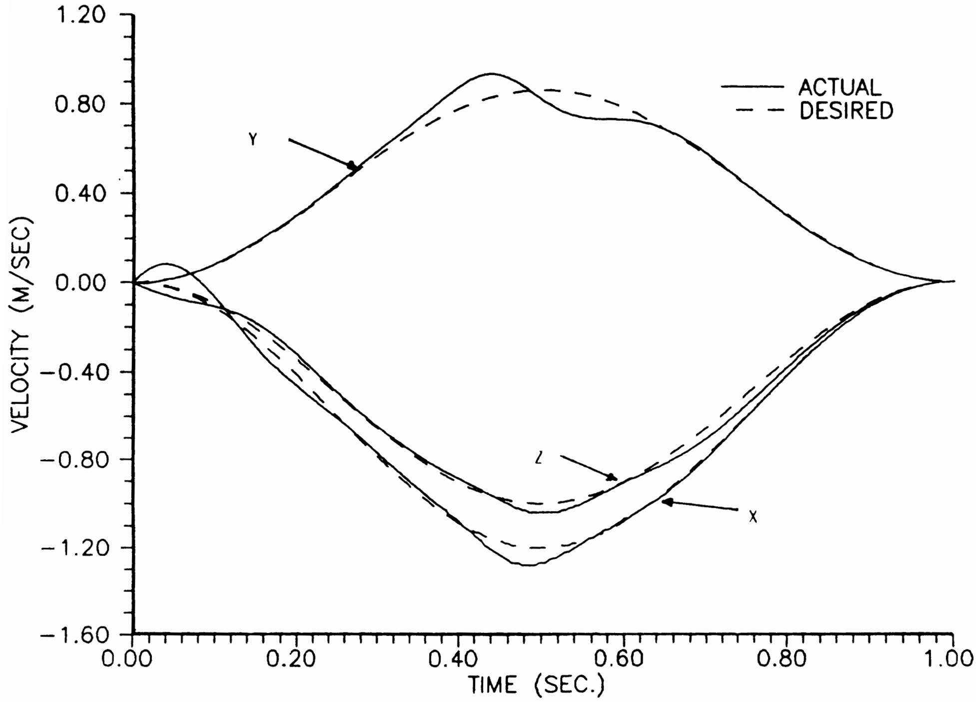

FIG. 3. The actual and desired velocity time history of the end effector point *E*.

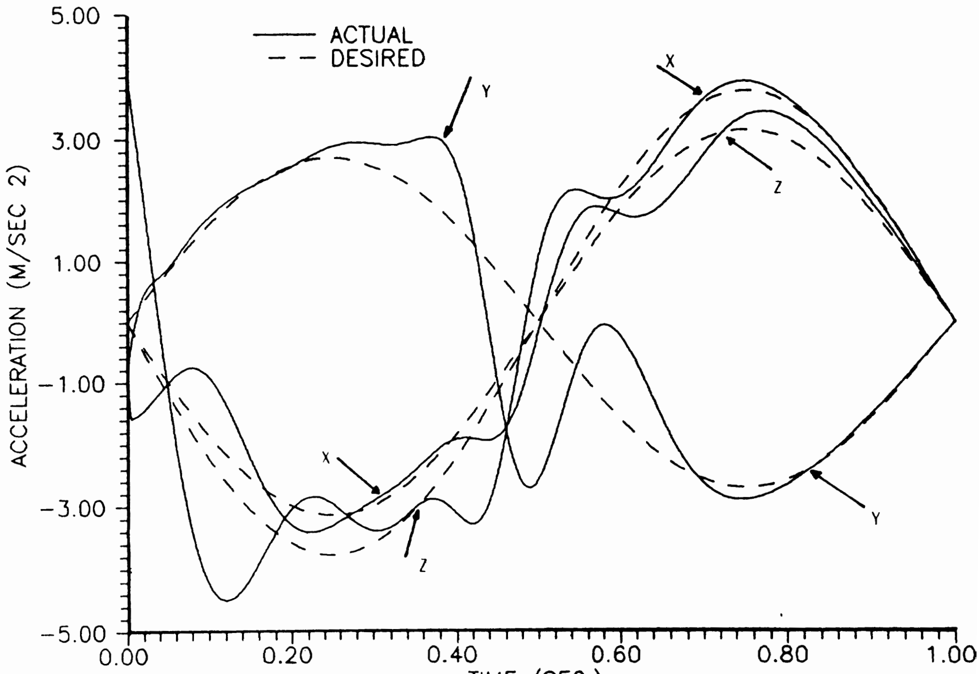

FIG.4. The actual and desired acceleration time history of the end effector point *E*.

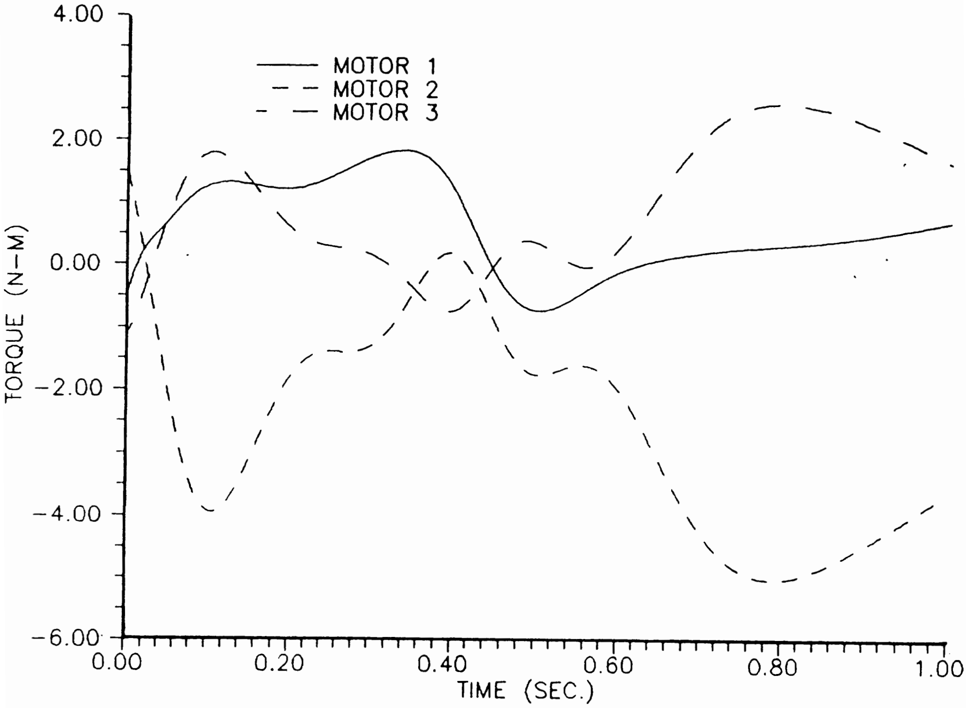

FIG.5. The time history of the three driving torques.

The performance of the control law as simulated by the modeling procedure gave a maximum error of 12 mm in the end effector position tracking. Depending on the application, this error might be acceptable. More accurate tracking could be obtained using more sophisticated control laws[15]. The purpose in this paper is just to demonstrate the utility of the proposed dynamic modeling procedure in predicting the dynamic behavior of multi-rigid body systems subjected to kinematic constraints.

5. Conclusion

This paper presented a dynamic modeling procedure using the virtual work principle. This procedure is suitable for modeling closed and open loop chains of rigid bodies. It generates for each rigid body a dynamic matrix and vector analogous to the element stiffness matrix and loading vector of the finite element method. The kinematic motion parameters are divided into dependent and independent coordinates in order to allow for flexibility in specifying the independent degrees of freedom of the system. The constraint equations between the two sets of coordinates are analogous to the boundary conditions in the finite element method. The equations of motion generated at the end of the dynamic modeling procedure are arranged so that they could be solved in the forward or inverse dynamic mode

The procedure is implemented in a general computer program. A numerical example is presented where a mixed loop robot is dynamically modeled and the features of the modeling procedure are employed. The equations of motion of the mixed loop robot example are solved in the forward dynamic mode where the driving torques are specified by the control law.

The dynamic modeling procedure presented in this paper is believed to be a good core for a general purpose dynamic modeling package. This package should include:

- A procedure for automatic generation of constraint equations and their derivatives
- A modeling procedure for introducing joint and body flexibility.
- A facility to include various types of friction loads at the joints.

References

[1] **Kane, T.R.** and **Levinson, D.A.,** *Dynamics: Theory and Applications*, McGraw-Hill Book Company, U.S.A., pp. 110-135 (1985).

[2] **Kane, T.R.** and **Levinson, D.A.,** The use of Kane's dynamical equations in robotics, *Intl. J. Robotics Res.*, **2**(3): 3-21 (1983).

[3] **Wang, J.T.,** Inverse dynamics of constrained multibody systems, *J. Appl. Mech. Trans. ASME*, **57**(3): 750-757 (1990).

[4] **Huston, R.L.,** Multibody dynamics formulation via Kane's equations, *AIAA Dynamics Specialist Conference, Long Beach, CA. USA, Apr 5-6 1990. Washington, DC, USA*, PP. 424-430 (1990).

[5] **Garcia, J., Unda, J.** and **Avello, A.,** Natural coordinates for the computer analysis of multibody systems, *Computer Methods in Applied Mechanics and Engineering*, **56:** 309-327 (1986).

[6] **Uicker, J.J.,** *On the Dynamic Analysis of Spatial Linkages Using 4 × 4 Matrices*, Ph.D. thesis, Northwestern University, U.S.A., pp. 30-60 (1965).

[7] **Paul, R.P.,** *Modeling Trajectory Calculation and Servoing of a Computer Controlled Arm, A.I. Memo 177, Stanford Artificial Intelligence Laboratory*, Stanford University, Sept. U.S.A. (1972).

[8] **Lewis, R.A.,** *Autonomous Manipulation on a Robot: Summary of Manipulator Software Functions, Technical Memo 33-679, Jet Propulsion Laboratory*, Pasadena, California (1974).

[9] **Agarwal, O.P.** and **Chung, S.L.,** Superelement model based on lagrangian coordinates for multibody system dynamics, *Comput. Struct.* **37**(6): 957-966 (1990).

[10] **Luh, J.Y.S., Walker, M.W.** and **Paul, R.P.,** On-Line Computational Scheme for Mechanical Manipulators, *Trans. ASME, J. Dynamic Systems, Measurements and Control*, **120:** 69-76 (1980).

[11] **Ginsberg, J.H.,** *Advanced Engineering Dynamics*, Harper & Row, Publishers, New York, pp. 286-298 (1988).

[12] **Denavit, J.** and **Hartenberg, R.S.,** A kinematic notation for lower pair mechanisms based on matrices, *ASME Journal of Applied Mechanics, June*, **77:** 215-221 (1955).

[13] **Goldenberg, A., Benhabib, B.** and **Fenton, R.,** A complete generalized solution of the inverse kinematics of robots, *IEEE Journal of Robotics and Automation*, **RA-1(1) March:** 14-20 (1985).

[14] **Craig, I.J.,** *Introduction to Robotics, Mechanics and Control*, Addison-Wesley Publishing Company, Reading, MA, U.S.A., pp. 230-245 (1986).

[15] **Koivo, A.,** Force-position-velocity control with self-tuning for robotic manipulators, *Proc. of 1986 International Conference on Robotics and Automation, San Francisco, California, April 1986*, pp. 1563-1568.

Appendix A

The homogeneous transformation matrix of Denavit and Hartenberg (D-H)[12] approach is given by

$$[A]_k^l = \begin{bmatrix} \cos\theta & -\sin\theta\cos\alpha & \sin\theta\sin\alpha & a\cos\theta \\ \sin\theta & \cos\theta\cos\alpha & -\cos\theta\sin\alpha & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & s \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(A-1)}$$

The transformation matrix $[A]_k^l$ transforms the coordinates of a point given in the $l-th$ coordinates system to the $k-th$ coordinate system. The parameters of this transformation matrix are defined in Fig. A-1.

The differential of the transformation matrix $[A]_k^l$ could be written in operator form as

$$d[A]_k^l = dq[B]_k^l\,[A]_k^l \tag{A-2}$$

where dq is the differential of the varying parameter in the transformation matrix which depends on the type of kinematic pair that exists between the two rigid bodies. The operator matrix $[B]_k^l$ is related to the $[\hat{B}]_{i_j}$ in a manner similar to that between $[A]_k^l$ and $[\hat{A}]_{i_j}$ as explained in Section 3.3. The matrix $[B]_k^l$ is defined as

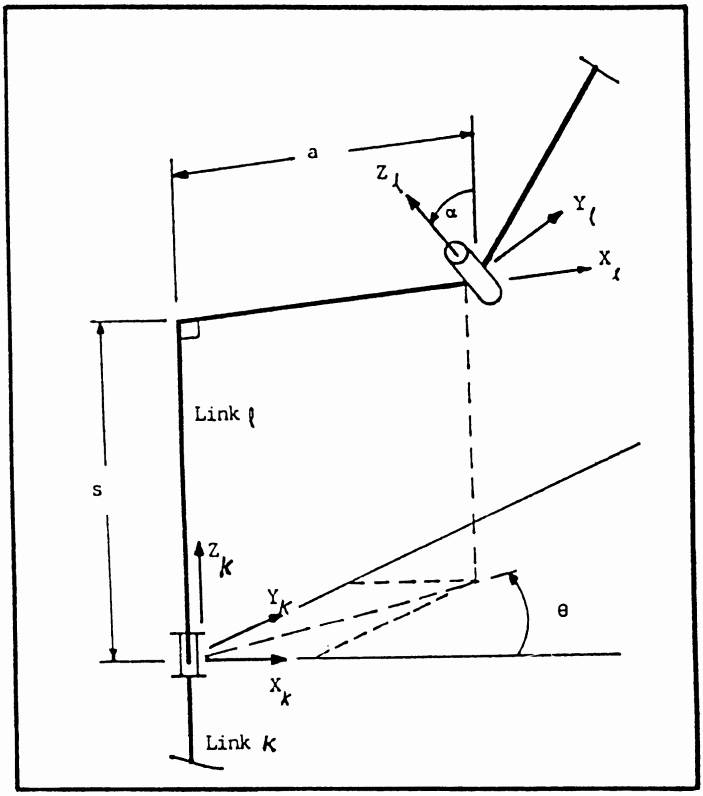

FIG. A-1. The homogeneous transformation matrix parameters.

$$[B]_k^l = \begin{cases} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \text{, for revolute pairs } (dq = d\theta). \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \text{, for prismatic pairs } (dq = ds). \end{cases} \tag{A-3}$$

Appendix B

The kinematic parameters for the six homogeneous transformation matrices of the mixed loop robot are given in Table B-1.

TABLE. B-1. The kinematic parameters of the system.

$k-l$	θ	s	a	α
0 - 1	q_1	L_1 = 30 cm	0	90°
1 - 2	q_2	0	L_2 = 30 cm	0°
2 - 3	q_5	0	L_3 = 30 cm	90°
1 - 4	q_3	0	L_4 = 15 cm	0°
4 - 5	q_6	0	L_5 = 26 cm	0°
5 - 6	q_4	0	L_4 = 15 cm	0°

The constraint equations for the mixed loop robot are given by

$$q_1 = \psi_1 / r_1 \tag{B-1}$$

$$q_2 = \left(\frac{1 + \frac{1}{r_2}}{r_1} \psi_1 - \frac{1}{r_2} \psi_2 \right) / r_2 \tag{B-2}$$

$$q_3 = \left(\frac{1 + \frac{1}{r_4}}{r_1} \psi_1 - \frac{1}{r_4} \psi_3 \right) / r_5 \tag{B-3}$$

$$q_4 = \pi - q_2 + \beta + q_3 \tag{B-4}$$

$$q_5 = q_{5_o} - r_6(q_4 - \gamma_o) \tag{B-5}$$

$$q_6 = q_2 - \beta - q_3 \tag{B-6}$$

where

- r_i are the gear ratios,
- β is the constant angle between the two sides of link 2, see Fig. 1, (β = 57°),
- q_{5_o} is the initial value of q_5 when the bodies are assembled (q_{5_o} = 46°), and
- γ_o is the initial value of the angle γ when the bodies are assembled (γ_o = 115°).

The gear ratios r_i are given by

$r_1 = N_2/N_1 = 8$, $r_2 = N_4/N_3 = 2$, $r_3 = N_8/N_7 = 8$,
$r_4 = (N_{10}N_{12}) / (N_9 N_{11}) = 2$, $r_5 = N_{14}/N_{13} = 8$, and $r_6 = N_{15}/N_{16} = 2$.

Where N_i is the number of teeth for gear G_i.

The inertial parameters for the six rigid bodies of the robot are given in Table B-2.

TABLE B-2. The inertial parameters of the system.

Body number	Mass (kg)	Center of mass (cm)			Mass moment of inertia (kg.m^2)		
		x	y	z	I_{xx}	I_{yy}	I_{zz}
1	2.0	0.0	-15.0	0.0	0.044	0.008	0.044
2	3.11	-12.5	- 4.3	0.0	0.003	0.013	0.016
3	0.22	-18.8	0.0	0.0	0.004	0.02	0.02
4	0.15	-10.0	0.0	0.0	0.01	0.015	0.02
5	0.06	-13.0	0.0	0.0	0.0	0.004	0.004
6	0.15	- 5.0	0.0	0.0	0.01	0.015	0.02

The diagonal elements used in the numerical example for the feedback gain matrices of the control law are:

$K_{p11} = 100.0$, $K_{p22} = 25.0$, $K_{p33} = 25.0$ N.m,
$K_{v11} = 1.0$, $K_{v22} = 0.5$, and $K_{v33} = 0.5$ N.m.s.

The inertial coordinates of the initial and terminal points of the spacial line tracked by the end effector of the robot are $P_i = (0.3, 0.0, 0.8)$ m and $P_t = (-0.3, 0.4, 0.3)$ m, respectively. The path was tracked within 1.0 second using 0.001 second integration time step. The end effector was carrying a mass of 5 kg at its tip.

طريقـة عامـة لتمثيـل ديناميـكا النـظم متعـددة الأجسام الصلبـة مع التطبيـق على ذراع آلي ثـلاثي المحـاور وذي نظـام قيـادة مثبت عـلى القـاعـدة

محمد ح. ف. دادو
قسم الهندسة الميكانيكية ، كلية الهندسة والتكنولوجيا ، جامعة الأردن
عمـــان - الأردن

المستخلص . تقدم هذه الورقة طريقة عامة لتمثيل السلوك الديناميكي لأنظمة متعددة الأجسام الصلبة ، وتحتوي على مقيدات حركية بين عناصرها المختلفة . ترتكز هذه الطريقة على مبدأ الشغل الافتراضي . تستعمل مجموعتان من المتغيرات لوصف حركة الأجسام ضمن النظام ، هاتان المجموعتان هما الإحداثيات المستقلة والإحداثيات غير المستقلة . وهناك مجموعة من معادلات التقييد تربط بين هاتين المجموعتين من الإحداثيات . يتم اشتقاق مبدأ الشغل الافتراضي للنظام بشكل كامل باستخدام الإحداثيات غير المستقلة . ويتم الربط بين التغيرات التفاضلية لمجموعتي الإحداثيات باستخدام معادلات التقييد للوصول إلى عدد منخفض من المعادلات التي تصف النظام ، وذلك باستخدام الإحداثيات المستقلة فقط . وتسهل هذه الطريقة عملية تمثيل أنظمة الأجسام المتعددة الكبيرة ، والتي تحتوي على مقيدات للحركة . وهذا التسهيل في الطريقة لا يؤدي إلى زيادة في عدد المعادلات التي تصف النظام الديناميكي . هناك مثال رقمي مفصل مقدم في هذه الورقة ، حيث تم التأكيد على ميزات هذه الطريقة . ويعرض هذا المثال ذراعًا آليًا فراغيًا يحتوي على مجموعة من الأجسام التي تشكل أنشوطة ، وهو يتميز بأن مولدات الحركة جميعها مثبتة على قاعدة .